The Superposition of Discrete-Time Markov Renewal Processes with an Application to Statistical Multiplexing of Bursty Traffic Sources

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Abstract

The main contribution in this paper is the introduction of a methodology for approximately characterizing the superposition process of $N \geq 2$ arbitrary (and possibly heterogeneous) discrete-time Markov Renewal Processes (MRP). In this model, the superposition process is characterized by a MRP with a state space that grows exponentially with N. We consider an on/off traffic source model, where the distribution of the on and off periods is arbitrary, as a special case of the general MRP. Subsequently, a queueing model for a FIFO finite-buffer multiplexer with arbitrary on/off input sources is analyzed. We provide numerical results for testing the algorithms introduced in the paper. We also study the effect of some of the statistical properties of on/off input sources on the multiplexer's performance.

1 Introduction

In this paper, we consider the problem of characterizing the superposition of multiple discrete-time Markov Renewal Processes (MRP). A lot of research has been reported in the area of characterizing the superposition process of renewal and Markovian processes. However, to the best of our knowledge characterizing the superposition of MRP's has not as yet been adequately addressed.

Non-Markovian processes are more versatile since they can provide realistic models of arrival processes in teletraffic analysis. However, the analysis of queueing systems with non-Markovian input is considerably more complex than that with renewal and Markovian arrival processes.

Cherry and Disney [1] studied the superposition of two continuous-time MRP's. The structure of the interval process resulting from superposing two independent MRP was characterized. The resulting stochastic process has a very large number of states which limits the applicability of the model to two processes.

Korolyuk [7] introduced a mathematical model for the superposition of multiple independent continuous-time MRP's. The model keeps track of the time each process spends in the current state. This limits the model's applicability to cases where analytic expressions for the sojourn times between states can be found. The author also introduced the notion of *phase space lumping*, an aggregation process, in order to reduce dimensionality of the MRP.

Following a similar methodology to the one presented in [1] and [7], we characterize the superposition of $N \geq 2$ independent discrete-time MRP's. The constructed superposition, according to our model, is a MRP with a state space that grows exponentially with N. It is to be noted here that in general the superposition of multiple MRP's is not necessarily an MRP. Hence the proposed methodology is an approximation because of the above reason in addition to other factors that will be described in detail in the remaining sections of the paper. Finally, a FIFO finite buffer statistical multiplexer with N arbitrary on/off arrival processes was analyzed. The analysis is one of the main contributions of the paper due to the new model of the superposition of the arrival processes considered.

Sohraby [9] presented innovative results on the tail behavior of a multiplexer with infinite waiting room shared by multiple arbitrary on/off sources. For the case of heterogeneous sources, the model is valid only in heavy traffic. For the case of homogeneous sources, the model can be used for all levels of multiplexer utilization. In the heterogeneous case, the asymptotic decay parameter is a simple function of the first two moments of the on and off periods and peak rates. However, our results show that for the case of a multiplexer with finite buffer, more parameters are needed than just the first two moments of the on and off period lengths (when peak rate is equal to link speed).

The rest of this paper is organized as follows. In section 2, we first provide a quick review of Markov renewal processes and present some results from discrete-time renewal theory. We then proceed to describe the methodology for characterizing the superposition of multiple independent MRP's. In section 2.3, we present the arbitrary on/off source model. In section 3, a finite buffer FIFO statistical multiplexer shared by multiple arbitrary on/off traffic sources is introduced and analyzed. In section 4, we present results of testing the algorithms and methods presented in this paper. In section 5, we examine the effects of some of the parameters of an arbitrary on/off source on the performance of a statistical multiplexer. Section 6 gives the conclusions of this paper.

2 The Superposition of Multiple Independent Discrete-time Markov Renewal Processes

2.1 Discrete-time Markov Renewal Processes and Some Basic Results from Renewal Theory

A Markov renewal process (MRP) is a stochastic process which moves from one state to another with a random sojourn times which has a distribution that depends on the state being visited as well as the next state to be entered. The successive states visited by the MRP form a Markov chain. Let Ξ be the state space of an MRP.

Definition: The stochastic process $(X,T) = \{X_r, T_r; r \in Z^+\}$ where $X_r \in \Xi$, $T_r = n\tau$, $n \in Z^+$, and $0 = T_0 \le T_1 \le T_2 \cdots$, is a discrete-time MRP with state space Ξ if

$$Pr\{X_{r+1} = x, T_{r+1} - T_r = k\tau | X_0, \cdots, X_r; T_0, \cdots, T_r\} = Pr\{X_{r+1} = x, T_{r+1} - T_r = k\tau | X_r\}$$

for all $r, k \in Z^+$ and $x \in \Xi$. Let us further assume that (X,T) is time-homogeneous; that is

$$P\{X_{n+1} = y, T_{n+1} - T_n = k | X_n = x\} = G(x,y,k)$$

independent of n. The family of probabilities $\boldsymbol{G} = \{G(x,y,k) : x, y \in \Xi, k \in Z^+\}$ is called a discrete-time semi-Markov kernel on Ξ. The process (X) is called the associated semi-Markov process (SMP) of the MRP (X,T). Whenever appropriate, we would use the term SMP in place of MRP.

The sum $p(x,y) = \sum_{k \ge 0} G(x,y,k)$ is not necessarily equal to one, but $p(x,y) \ge 0$ and $\sum_{y \in \Xi} p(x,y)$ must be equal to one. The $p(x,y)$ are in fact the transition probabilities for some Markov Chain with state space Ξ and probability transition matrix $\boldsymbol{P} = [p(x,y)]$.

We now review some basic results from renewal theory that will be used below to construct the superposition process. Let $f(k)$, $0 \le f(k) \le 1$, $k \ge 0$, be a probability mass function. Let $\omega = \sum_{k=0}^{\infty} f(k)$, $0 < \omega \le 1$, and $\bar{M} = \sum_{k=0}^{\infty} k f(k)$. Let $F(k) = \sum_{l=0}^{k} f(l)$, be the associated cumulative probability density function of length up to k. The probability mass function of the residual life-time is given by :

$$\hat{f}(k) = \omega \left[\frac{\omega - F(k-1)}{\bar{M}} \right], \quad k = 1, 2, \cdots \tag{1}$$

Note that when ω is equal to 1, we get the known results for discrete-time renewal theory [10].

2.2 Characterization of The Superposition Process of Multiple Independent Markov Renewal Processes

Consider $N \geq 2$ independent discrete-time MRP's. Each individual MRP i is characterized in terms of a semi-Markov kernel $\boldsymbol{G}_i = [g_i(x, y, k)]$ defined over the set of states $1, 2, \cdots N_i$, $N_i \geq 1$. In order to characterize the superposition process, we have to define the states of the process and then for each state find the distribution of the sojourn time between the state and any other directly accessible state of the superposition process.

In [1] the superposition state descriptor is a vector consisting of the current state of each component process and the time each process has spent in its current state since its last transition. It is clear that such a characterization results in an enormous state space, since the time a component process spends in a state can be quite large especially when some of the functions $g_i(x, y, k)$ have a long tail.

To show how our superposition is constructed, let us assume for the moment that only one particular process, say process i, has made a state transition and another process j, $j \neq i$, did not change state at that instant and that it is in state x_j. Let $T(x_j)$ be the time that process j has spent so far in state x_j. Due to the independence of the two processes, the distribution of $T(x_j)$ will be equal to the distribution of the life-time in state x_j. The time $T'(x_j)$ that it takes process j to undergo a state change would have a distribution equal to the residual life-time distribution at state x_j.

This is illustrated through the example given in figure 1. At the instant marked *observation instant*, process 1 makes a state transition from state 2 to state 3. The distribution of time until processes 2 and 3 experience a state transition, $T'(x_2)$ and $T'(x_3)$ respectively, is approximately equal to the residual life-time distribution in state 2 and state 1 of each process respectively. This is true when we take all possible realization of the above event and assuming that all processes are independent.

We define the superposition state at instants when one or more of the individual processes experience a state transition. Note that in discrete-time systems, it is possible that one or more processes change state at the same instant. The superposition state is described by the tuple $[(x_1, t_1), (x_2, t_2), \cdots, (x_N, t_N)]$ where $x_i \in \{1, 2, \cdots, N_i\}$ is the state of process i observed immediately after a transition occurs, and $t_i \in \{0, 1\}$ indicates whether process i has changed state or not, with $t_i = 1$ iff process i has changed state.

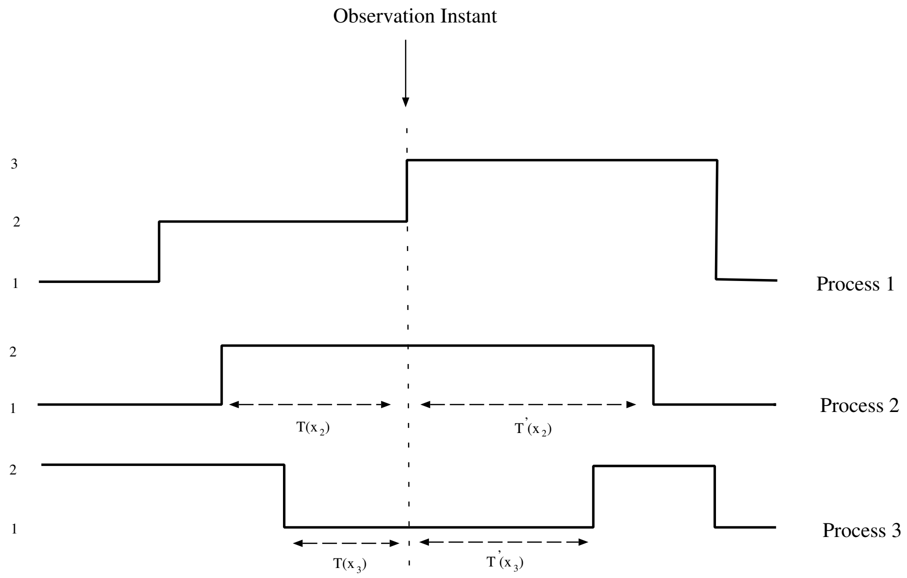


Figure 1: The distribution of residual life-time for a process not experiencing a state change.

The state space Ξ of the superposition process is given by

$$\Xi = \left\{ [(x_1, t_1), (x_2, t_2), \cdots, (x_N, t_N)] : x_i \in \{1, 2, \cdots, N_i\},\ t_i \in \{0, 1\},\ \sum_{i=1}^{N} t_i \neq 0 \right\}$$

Note that $\sum_{i=1}^{N} t_i \neq 0$ because the superposition process is observed immediately after a transition occurs. The number of states in the superposition process is obviously equal to $(\prod_{i=1}^{N} N_i)(2^N - 1)$.

In order to fully characterize the superposition process as a MRP, we must obtain the distribution of the time it takes the MRP to move from a state u to state v, where $u, v \in \Xi$ and v is directly accessible from u. Let the probability of going from u to v in k slots be denoted by $q(u, v, k)$. Then $\boldsymbol{Q} = \left[q(u, v, k)\right]$ is the semi-Markov kernel of the superposition process. We now proceed to calculate the functions $q(u, v, k)$.

Let $\hat{g}_i(x, y, k)$ be the residual life-time probability mass function associated with $g_i(x, y, k)$ which is calculated using equation 1. Also, let $G_i(x, k) = \sum_{y=1}^{N_i} \sum_{l=1}^{k} g_i(x, y, l)$, be the cumulative distribution function of sojourn time up to slot k in state x for process i, and $\hat{G}_i(x, k) = \sum_{y=1}^{N_i} \sum_{l=1}^{k} \hat{g}_i(x, y, l)$ be similarly defined for the functions $\hat{g}_i(x, y, k)$.

The function $q(u, v, k)$ depends on the values of $(x_i(u), t_i(u))$ and $(x_i(v), t_i(v))$ for all processes i, where $(x_i(s), t_i(s))$ is the state of process i when the superposition is in state s. We assume that the transition from state u to state v would take place in k slots and then proceed to calculate the probability of this event to occur. Consider an arbitrary process i, based on the values of $t_i(u)$ and

$t_i(v)$, the following four distinct cases are possible:

I- $t_i(u) = t_i(v) = 0$. In this case we have $x_i(u) = x_i(v)$, i.e. process i does not change state in either state u or v. The probability of such an event to occur for process i is approximately given by $\phi_i(u, v, k) = 1 - \hat{G}_i(x_i(u), k)$, which is the probability that the residual life-time in state $x_i(u)$ is greater than k.

II- $t_i(u) = 0$ and $t_i(v) = 1$. In this case process i changes state at v but not at u. Since process i has not changed state at u, the probability that this event occurs is $\phi_i(u, v, k) = \hat{g}_i(x_i(u), x_i(v), k)$, which is the probability that the residual life-time in state $x_i(u)$ is equal to k.

III- $t_i(u) = 1$ and $t_i(v) = 0$. In this case process i changes state at u but not at v. The probability that this event occurs is $\phi_i(u, v, k) = 1 - G_i(x_i(u), k)$, which is the probability that the sojourn time in state $x_i(u)$ is longer than k.

IV- $t_i(u) = t_i(v) = 1$. In this case process i changes state at both u and v. The probability that this event occurs is given by $\phi_i(u, v, k) = g_i(x_i(u), x_i(v), k)$, which is the probability that the sojourn time from state $x_i(u)$ to state $x_i(v)$ is equal to k slots.

Thus, due to the independence of the component processes, we have the result

$$q(u, v, k) = \prod_{i=1}^{N} \phi_i(u, v, k) \tag{2}$$

for all $u, v \in \Xi$ and $k \in Z^+$.

Another possible approximation method would be to assume that only one process may make a change at any state. Therefore at each state we have N ways out, all other changes would lead to the same state. This should reduce the computational complexity of the alogorithm used to construct the superposition process. We now discuss the computational complexity of the algorithm for constructing the kernel of the superposition process. Let us assume that the maximum length of the probability density function tail in any of the kernels $\{\boldsymbol{G}_i\}_{i=1}^{N}$ is given by L. The storage space needed for the semi-Markov kernel $\boldsymbol{Q}$ to be generated is $O\left((\prod_{i=1}^{N} N_i)^2 \, (2^N - 1)^2 L\right)$. The computational complexity (number of floating point multiplications) is $O\left((\prod_{i=1}^{N} N_i) \, (2^N - 1)^2 L\right)$. This is due to the fact that for an arbitrary state u, the number of states directly accessible from u after one transition is $(2^N - 1)$. Even for a small value of N, say $N = 5$, $N_i = 2 \, \forall i$, and $L = 1000$, the storage needed is $O\left(2^{20} \times 1000 \times L\right)$ which is excessively large for implementation on a conventional computer. By using sparse matrix techniques the storage capacity needed can be reduced to $O\left((\prod_{i=1}^{N} N_i) \, (2^N - 1)^2 L\right)$. Clearly, for a large number of processes, an approximate solution should be sought instead. For example, the MRP may be approximated by a Markov chain or by a renewal process.

A testing of the methodology presented above for characterizing the superposition process is given in section 4.1.

2.3 The Superposition of Multiple Arbitrary on/off Sources

In this section, we discuss the arbitrary on/off traffic source model which is to be used in the rest of the paper. Consider a traffic source that alternates between active (on) and idle (off) periods. The source always transmits one cell per slot when it is in the on state. For traffic source i, the length of the on and off periods has an arbitrary distribution. For source i, let $f_i^{on}(k)$ and $f_i^{off}(k)$ be the probability that the length of an on and off period is k time slots respectively. The stochastic process describing the source is in fact an alternating renewal process with two states which can be described by means of a MRP. Let the two states be 0 and 1 corresponding to the off and on state of the source respectively. The associated semi-Markov kernel is given by

$$\boldsymbol{G}_i(k) = \begin{bmatrix} 0 & f_i^{off}(k) \\ f_i^{on}(k) & 0 \end{bmatrix}.$$

We can now directly apply the results of section 2.2 to characterize the superposition process of $N \geq 2$ arbitrary on/off sources. Since $N_i = 2$, $i = 1, 2, \cdots N$, and $x_i \in \{0, 1\}$, the expression for $\phi_i(u, v, k)$ in equation 2 is simplified. For N sources, the superposition process has $2^N(2^N - 1)$ states.

3 Analysis of a Statistical Multiplexer with Multiple Arbitrary on/off Input Sources

Consider a FIFO finite buffer multiplexer serving $N \geq 2$ arbitrary on/off sources. Each source is described in terms of the probability density function of its on and off periods. A source emits cells at each time slot when it is in the on state. The multiplexer has $S \geq 1$ servers and can accommodate a total of $B \geq S$ cells at any time instant including those in service. The service time for all cells is constant and is equal to one time slot. The multiplexer can serve S cells every time slot. We assume that $N > S$, otherwise no queue will ever form in the multiplexer and the problem will be trivial to handle.

We seek the steady state probabilities, $\pi(n)$, $0 \leq n \leq B - S$, that there are n cells in the

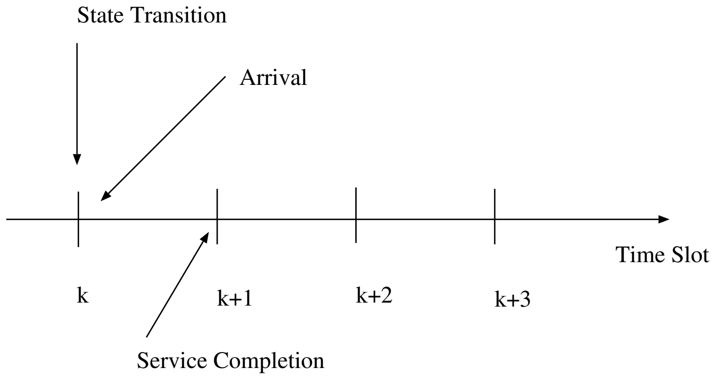


Figure 2: Timing of events in the early arrival model.

multiplexer's queue. From this we can obtain other measures of interest such as the mean queue length, the probability of full buffer and the cell loss probability.

Let us first discuss the timing of events in our system. We follow an *early arrival* timing model as defined by Hunt [6]. That is, during an arbitrary time slot, the sequence of events is as follows: a state transition in the superposition may occur, followed immediately by cell arrivals (if any), which is followed by service of waiting cells if there are any, and finally departure of cells that received service. This is shown in figure 2. If an arbitrary cell sees one or more empty servers upon its arrival, it is immediately admitted to one of the available servers without waiting till the next slot. Hence, the multiplexer is effectively of the cut-through type.

The superposition process of the N sources is first characterized as a MRP. Let M be the number of states of the MRP, with states numbered $0, 1, \cdots, M-1$ and $\boldsymbol{Q} = [q(x, y, k)]$ be its semi-Markov kernel. Let $a(h)$ be the number of active sources at state $h \in \{0, 1, \cdots, M-1\}$ of the superposition process.

System State: The system state at any particular slot is described by the pair (n, h) where $0 \leq n \leq B - S$ is the number of cells in the multiplexer buffer at the beginning of the slot (not counting the cells that are to arrive at this slot) and $0 \leq h \leq M-1$ is the current state of the superposition process. We note here that the process (n, h) at successive time slots does not form a Markov chain. This is due to the non-Markovian nature of the superposition process. However, by observing the system at instants immediately after the superposition process experiences a state transition, the states (n, h) at these instants form an embedded Markov chain since successive states visited by a MRP form a Markov chain.

Solution Method: The probability transition matrix $\boldsymbol{P}$ governing transitions between all possible states (n, h) is first generated. Then the embedded steady state probabilities $\hat{\pi}(n, h)$ are calcu-

◇ Let $p[(n,h),(n^{'},h^{'})] = 0$ for all states.
◇ For all states (n_0, h) do
  ⋆ For all values of k and $h^{'}$
  ⋆ Find n_k using equation 3
  ⋆ If $q(h,h^{'},k) \neq 0$ then let $p[(n_0,h),(n_k,h^{'})] = p[(n_0,h),(n_k,h^{'})] + q(h,h^{'},k)$

Figure 3: The algorithm for generating the probability transition matrix.

lated. Finally, the arbitrary point probabilities of observing state (n,h), $\pi(n,h)$, are obtained from $\hat{\pi}(n,h)$. The two fundamental technical difficulties that arise here are generating the matrix P and the calculation of $\pi(n,h)$ from $\hat{\pi}(n,h)$.

Generation of the Probability Transition Matrix: Consider the queue occupancy evolution process at the multiplexer. Assume that the superposition has just made a transition to state h and that the number of cells in the multiplexer immediately before that transition occurs was n_0. During the time interval at which the superposition process is in state h, $a(h)$ cells arrive at the beginning of each slot. At each time slot, if the number of newly arrived cells plus the number already in the system is greater than B, then the excess cells are dropped randomly, i.e. independently of which source they originate from. By the end of a time slot a maximum of S cells in the multiplexer (possibly including those who have just arrived) are served. Assume that the superposition is in state h and that it makes a transition to state $h^{'}$ in k slots. Also, assume that the number of cells in the multiplexer when the superposition process made the transition to state h was n_0. Then, the number of cells in the multiplexer after r slots can be calculated using the following recursive equation:

$$n_r = max\left(0, min(n_{r-1} + a(h), B) - S\right), \; r = 1, 2, \cdots \tag{3}$$

By applying the above equation k times we can find the number of cells in the multiplexer k slots after a transition to state h occurs. By conditioning on the probability that superposition makes a transition from state h to state $h^{'}$ in k steps, we *increment* the probability of going from state (n_0, h) to state $(n_k, h^{'})$ by $q(h,h^{'},k)$. Note that it is possible that a specific state $(n', h^{'})$ can be visited from state (n_0, h) for different values of k. For example if $n_0 = 0$ and $a(h) = 0$, then for all $k \geq 1$, state $(0, h^{'})$ will be the next visited state of the embedded Markov chain. This suggest the algorithm in figure 3 for generating the probability transition matrix

Once the probability transition matrix $\boldsymbol{P}$ is generated, we solve for the invariant probability vector $\hat{\pi}(n,h)$ which is the probability of observing the queueing system in state (n,h) given that the superposition process has just undergone a state transition.

Arbitrary-time Probability Calculation: The key to the calculation of the arbitrary-time probability distribution of the queue occupancy is that the system evolution is deterministic given a specific state h of the superposition process, an initial queue occupancy level n_0, and the number of slots k measured from the instant when the superposition process moved to state h.

Let the state of the system at an instant where a transition occurs be (n_0, h). Let us assume that the superposition process makes a transition to state h' after $k \geq 0$ slots with probability $q(h, h', k)$. Then, all states (n_r, h), $1 \leq r \leq k-1$, where n_r is calculated using equation 3, will be observed with probability one, conditioned on the initial state (n_0, h) and that a transition from state h to state h' occurs in $l > k$ slots. Probabilities $\pi(n, h)$ can then be calculated using the algorithm shown in figure 4. Note the essential normalization step.

◇ For all states (n, h), let $\pi(n, h) = 0$
◇ For all states (n_0, h) do
◇ For all possible states h'
    For all possible values of l
        If $q(h, h^{'}, l) \neq 0$ then
            for all values of k, $0 < k < l$
                Find n_k from equation 3
                Let $\pi(n_k, h) = \pi(n_k, h) + \hat{\pi}(n_0, h)\, q(h, h^{'}, k)$
◇ Let $\kappa = \sum_{(n,h)} \pi(n, h)$
◇ For all states (n, h), let $\pi(n, h) = \frac{\pi(n,h)}{\kappa}$ `(Normalization)`

Figure 4: The algorithm for calculating the arbitrary-time probability

Once the arbitrary point probabilities $\pi(n, h)$ have been found, performance measures of the multiplexer like the mean queue length and the cell loss probability can be obtained. The mean queue length can be easily obtained from the probabilities $\pi(n, h)$. The probability of loss $P(Loss)$ is calculated as follows:

$$P(Loss) = \frac{\sum_n \sum_h min(n + a(h) - B, 0)\pi(n, h)}{\sum_n \sum_h a(h)\pi(n, h)} \tag{4}$$

which is equal to the average loss rate divided by the average arrival rate.

4 Testing and Evaluation of the Algorithms

In this section, numerical results are presented in order to test the algorithms presented in sections 2.2 and 3. The accuracy of these algorithms is assessed by comparing their results with detailed simulation results. First, we present results for the approximation method for characterizing the superposition process of discrete-time MRP's presented in section 2.2. Numerical results for testing of the statistical multiplexer model follows. The testing of these two algorithms is carried out assuming two MRP's or two on/off sources.

4.1 Testing of the Approximate Superposition of Discrete-time Markov Renewal Processes

In this section, we report results for testing the accuracy of the approximate characterization of the superposition process of multiple discrete-time MRP described in section 2.2. We compare the stochastic process obtained from characterizing the superposition of multiple MRP's using the proposed method and that obtained by simulation. This can be done by looking at the resulting semi-Markov kernel of the approximate and the simulated processes.

We consider the superposition of two MRP's with the following structure. Let $g(i,k)$ be the probability that the sojourn time in state i is k slots long, and let $p(i,j)$ be the underlying probability of going from state i to state j. Then, $g(i,j,k)$ is equal to $g(i,k)p(i,j)$, where $g(i,j,k)$ is the probability that the sojourn time from state i to state j is of length k. The first MRP has three states, the distribution of the sojourn time in each state is truncated geometric with the parameters shown in table 1. The underlying probability transition matrix is also shown in table 1. The second process has four states, the distribution of the sojourn time in states 0 and 2 is uniform and in states 1 and 3 is modified binomial. The parameters of the distributions and the underlying probability transition matrix are shown in table 2. In both tables 1 and 2, the column marked $Prob(Len = k)$ gives the probability that the sojourn time in a particular state is equal to k.

The approximate superposition process was obtained as described in section 2.2 and it is characterized by a MRP with 36 states. In order to test the superposition process we conducted a long simulation experiment and observed the statistical properties of the superposition process. In the results shown below, the relative error for a measured quantity x is defined as $|\frac{x(analytic)-x(simulation)}{x(analytic)}| \times 100$, where $x(analytic)$ and $x(simulation)$ is the estimate of x as obtained by analysis and simulation respectively.

The embedded probability of visiting a particular state and the associated relative error is shown in figure 5, the largest error is 0.53%. We also show the mean sojourn time in states of the superposition and the associated relative error in figure 6. The largest relative error is approximately 5.7%. The results in figure 7 show the distribution of sojourn time between two arbitrary source-destination pairs. The first pair is [(1,1),(1,0)] and [(1,0),(2,1)], while the second pair is [(0,0),(0,1)] and [(2,1),(0,0)]. The analytic and simulation results are almost identical.

The approximation model as described in section 2.2 was observed, in general, to be accurate. as we have experimented with other types of distributions and different processes and found the accuracy to be reasonable. Also, we observed that the more the states' sojourn time distribution resembles a memoryless distributions, the better the approximation is. This is due to the fact that if all the sojourn time distributions are memoryless the approximate superposition is actually an exact description of the actual superposition process.

State	$Prob(Len = k)$, $k = 1 \cdots L$
0	$(1-\alpha)\alpha^{k-1}/(1-\alpha^L)$, $\alpha = 0.998, L = 1000$
1	$(1-\alpha)\alpha^{k-1}/(1-\alpha^L)$, $\alpha = 0.995, L = 1000$
2	$(1-\alpha)\alpha^{k-1}/(1-\alpha^L)$, $\alpha = 0.99, L = 200$

(a) Sojourn time distribution.

$$\begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.0 & 0.3 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

(b) Prob. transition matrix.

Table 1: Parameters for the first MRP.

State	$Prob(Len = k)$, $k = 1 \cdots L$
0	$1/L$, $L = 800$
1	$\binom{L-1}{k-1}\ \alpha^{k-1}(1-\alpha)^{L-k}$, $L = 100, \alpha = 0.7$
2	$1/L$, $L = 300$
3	$\binom{L-1}{k-1}\ \alpha^{k-1}(1-\alpha)^{L-k}$, $L = 50, \alpha = 0.3$

(a) Sojourn time distribution.

$$\begin{bmatrix} 0.1 & 0.4 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.3 & 0.2 \\ 0.2 & 0.3 & 0.3 & 0.2 \end{bmatrix}$$

(b) Prob. transition matrix.

Table 2: Parameters for the second MRP.

4.2 Testing of the Statistical Multiplexer Analysis

In section 3 we presented a stochastic queueing model for the analysis of a finite buffer FIFO statistical multiplexer whose input is a MRP which can be the superposition of a number of independent arbitrary on/off sources. We report results from two experiments that have been carried out. In the two experiments the buffer size B was taken to be 40 and there was only one server ($S = 1$) in the multiplexer. In the graphs showing the obtained results, the curves labeled as *analysis* and *simulation* demonstrate the results obtained by characterizing the superposition process in terms of a 12-state MRP obtained using the results from section 2.2 and performing a queueing analysis as described in section 3 and by discrete-event simulation respectively. We note here that confidence

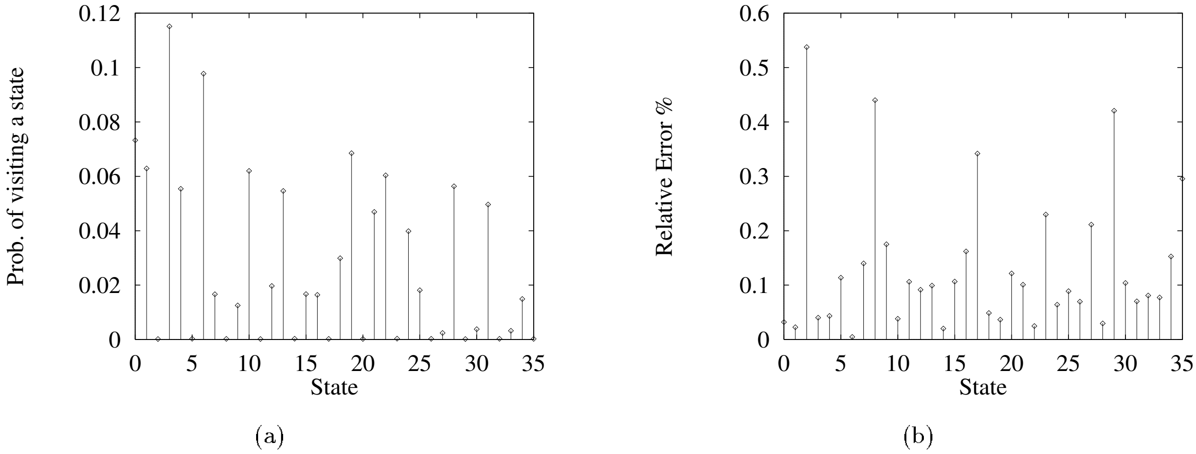


Figure 5: The embedded probability of visiting one of the states and the error relative to simulation results.

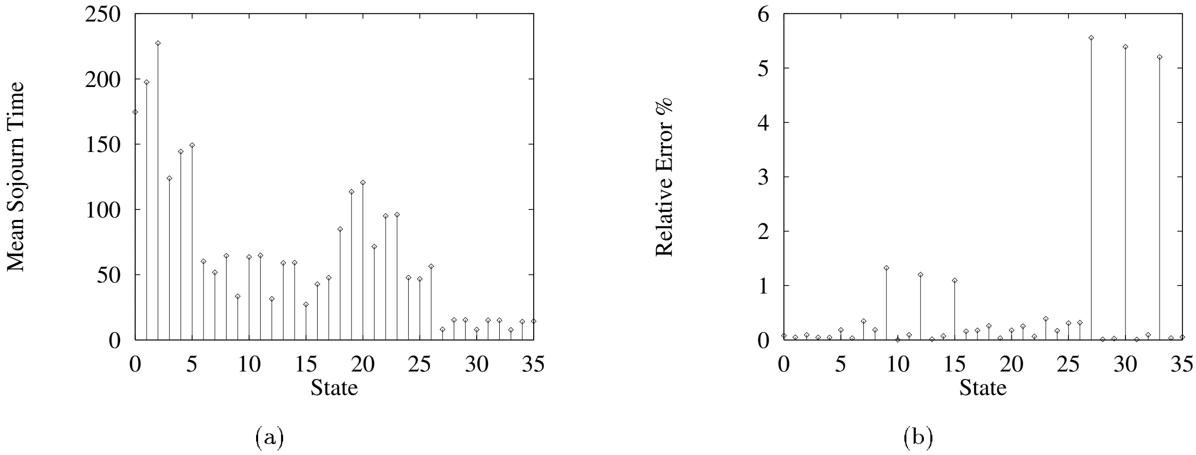
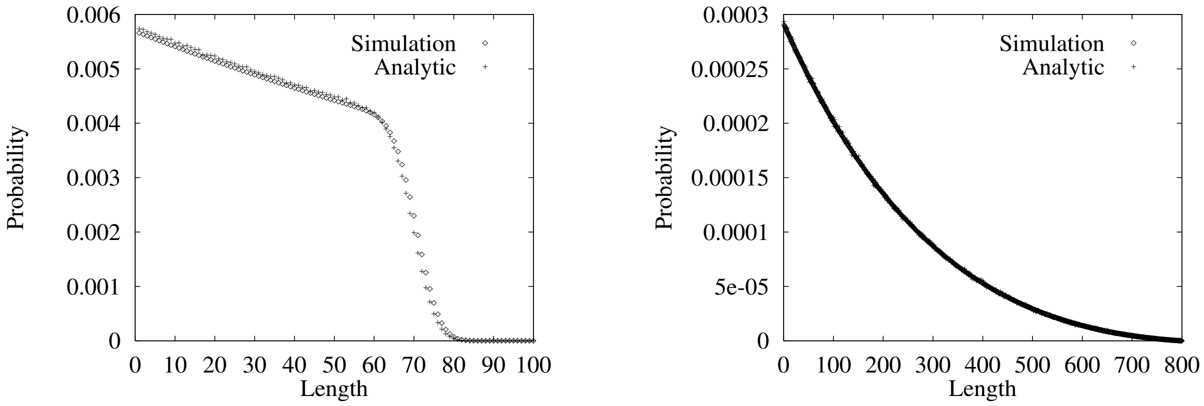


Figure 6: The mean sojourn time in a state and the error relative to simulation results.

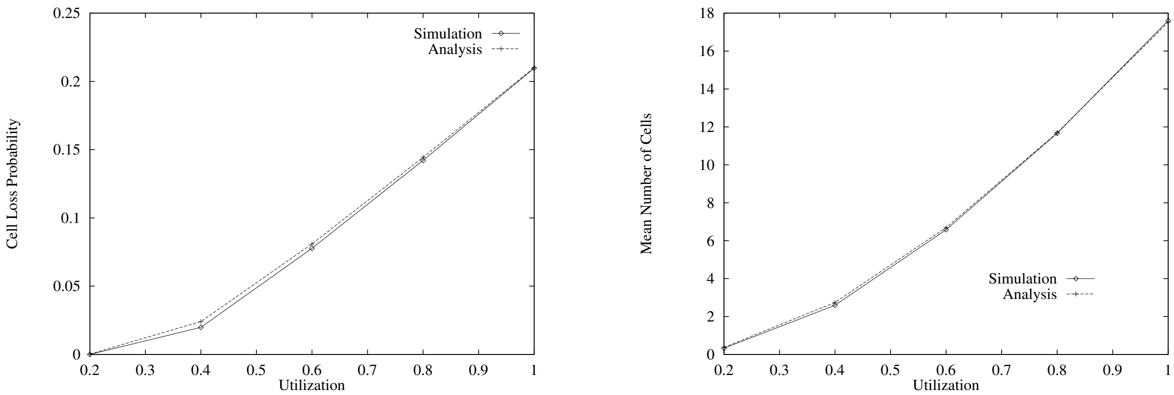


(a) $((1,1),(1,0)) \rightarrow ((1,0),(2,1))$ (b) $((0,0),(0,1)) \rightarrow ((2,1),(0,0))$

Figure 7: The distribution of the sojourn time between two arbitrary states.

intervals for the simulation results are not shown because they were very narrow. This is also true for the curves shown in the remaining sections.

Example 1: In the first experiment, we let the multiplexer input be two homogeneous on/off sources, where the on period is deterministic and the off period is mixture-of-geometric with mean lenght 251 and coefficient of variation to 1.5. The constant on period length is taken from the set $\{28, 63, 108, 167, 251\}$ providing a source average rate of $\{0.1, 0.2, 0.3, 0.4, 0.5\}$ respectively. Figure 8 shows the results for this particular model. We observe the proximity of the results obtained by simulation and detailed superposition.



(a) (b)

Figure 8: Testing of statistical multiplexer model. Buffer Size = 40, two input on/off sources with deterministic on and mixture-of-geometric off periods. (a) Cell loss probability, (b) Mean number of cells.

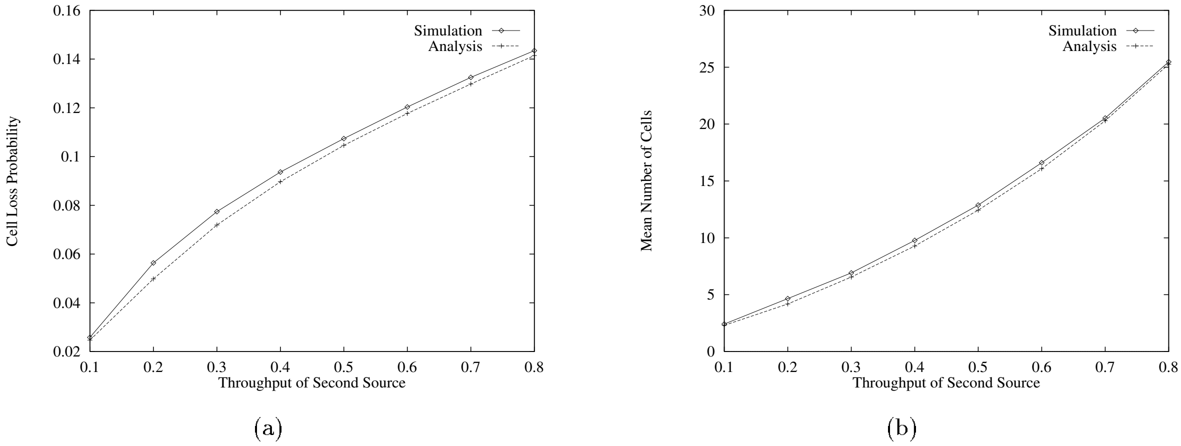


Figure 9: Testing of statistical multiplexer model. Buffer Size = 40, two heterogeneous input on/off sources with mixture-of-geometric on and off periods. (a) Cell loss probability, (b) Mean number of cells.

Example 2 - Heterogenous Sources: We tested the effect of source heterogeneity on the accuracy of the queueing model by studying the case of two *heterogeneous* on/off sources with mixture-of-geometric distribution for the on and off periods with the parameters shown in table 3(a) and table 3(b). For the second source the mean on period $\bar{on}$ is adjusted such that the source average rate takes values from the set $\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. The average rate of the first source is fixed to 0.2.

In accordance with the results of the previous experiments, we note here that the analysis and simulation results are pretty close. The results are shown in figure 9.

	Mean	*CV2*
On period	62.5000	5.0
Off period	250	1.5

(a) First source parameters

	Mean	*CV2*
On period	$\bar{on}$	5.0
Off period	250	1.5

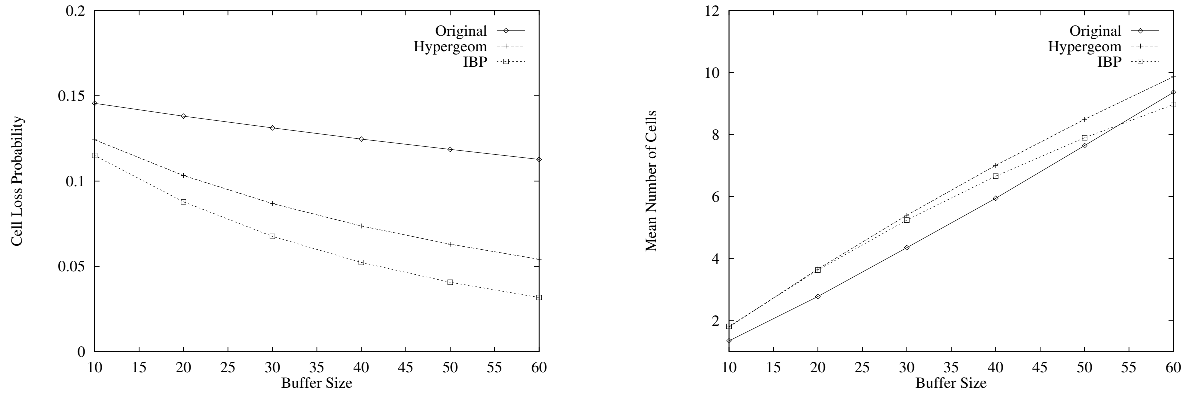
(b) Second source parameters

Table 3: Parameters for example 2.

5 Study of the Effect of Traffic Parameters on Queueing Performance

In this section, we conduct a study of the effect of various traffic parameters on the performance of the statistical multiplexer.

5.1 Effect of the Distribution of the On and Off Periods on the Multiplexer's Performance



(a) (b)

Figure 10: Effect of the distribution of the on and off periods on the multiplexer's performance. (a) Cell loss probability, (b) Mean number of cells.

In this section, we study the effect of the distribution of the on and off periods on the multiplexer's behavior. In the literature, it is common to approximate the distribution of sojourn times in a state by a geometric or a mixture-of-geometric distribution. The geometric distribution characterization requires only the first moment, while the mixture-of-geometric distribution characterization requires the first two moments. The question that usually arises is whether these approximations are accurate. We consider an extreme case where the original input source is correlated and then study how the multiplexer's performance would change if the source is replaced by an IBP (sojorun times are geometric) or by a source with mixture-of-geometric on and off periods. Note that in these two models the inter-arrival times are uncorrelated. We consider a single server multiplexer with two homogeneous input sources and a buffer size taken from the set $\{10, 20, 30, 40, 50, 60\}$.

The distribution of the on and off periods of the original input source is a mixture of two deterministic distributions. The inter-arrival time lag-1 correlation is equal to 0.048. The parameters of the pdf of the on and off periods are shown in table 4. The average rate of a single source is equal to 0.296, the mean on and mean off periods are 50 and 119 respectively, and the CV_{on}^2 and the CV_{off}^2 are 2.332 and 13.150 respectively. Given these values we can approximate the original source by an IBP source and a source with a mixture-of-geometric on and mixture-of-geometric off periods.

On period		Off period	
Length	Prob.	Length	Prob.
1	0.708333	20	0.95
169	0.291667	2000	0.05

Table 4: Parameters of the pdf of the on and off periods.

In figure 10 we plot the mean number of cells and cell loss probability for the original source, the single-moment approximation by an IBP source, and the two-moment approximation by a source with the mixture-of-geomtric on and off periods. The IBP model underestimates the cell loss probability and overestimates the mean number of cells (except when the buffer size increases above 52). This suggests that the IBP is not a faithful model for the original source of table 4.

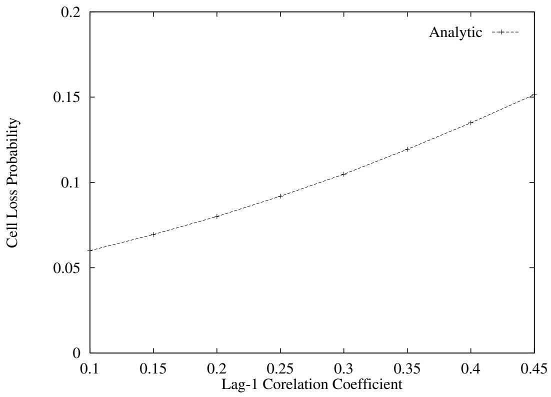
Sohraby [9] introduced a model for handling general on/off sources. The model gives an approximate upper bound for the cell loss probability as a function of the first two moments of the on and off periods assuming multiplexer with an infinite buffer size. This suggests that only the first two moments of the one an off periods affect the performance. However, the results in figure 10, demonstrate the inaccuracy of the two-moment approximation when the buffer size is finite. The approximate source model with two-moments matching provides an underestimation of the cell loss probability and mean number of cells. This shows that the two-moment approximation may not be accurate in all cases.

5.2 Effect of the Interarrival Time Correlation of Traffic Sources on the Multiplexer's Performance

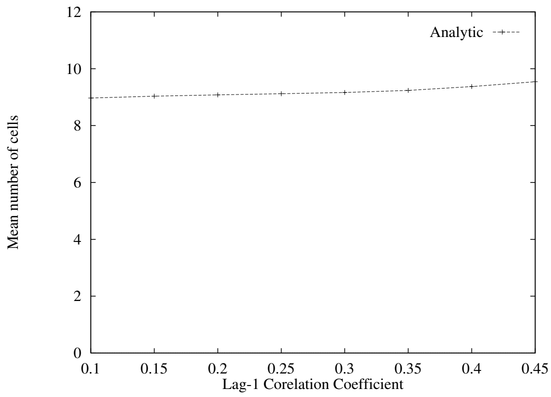
For an arbitrary on/off source as described in section 2.3, the lag-1 correlation coefficient of the inter-arrival time is given by Galmés, Perros, and Puigjaner [5]:

$$\phi_1 = \frac{f_{on}(1) - \frac{1}{\bar{on}}}{1 + CV_{off}^2 - \frac{1}{\bar{on}}}$$

where $f_{on}(1)$ is the probability that the on period is of length 1, $\bar{on}$ is the mean on period, and CV_{off}^2 is the squared coefficient of variation of the off period length (see Appendix A for the derivation of the result). We have been able to identify some distributions for which the value of ϕ_1 is non-negligible.



(a)



(b)

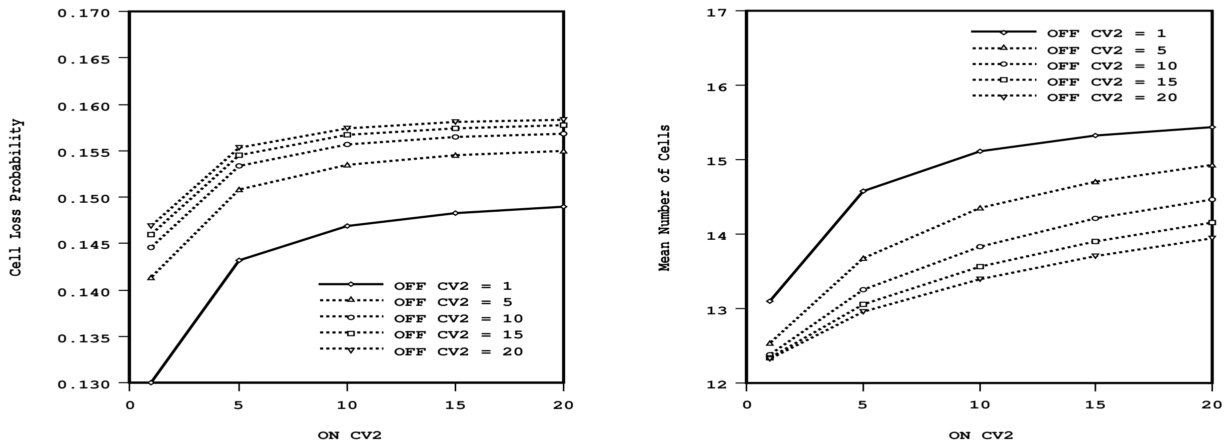
Figure 11: Effect of correlation on the multiplexer's performance. Buffer Size = 40, two input on/off sources with mixture-of-geometric on and off periods. (a) Cell loss probability, (b) Mean number of cells.

The key to obtaining such distributions is to concentrate a large portion of the probability mass at length 1 of the on period distribution, i.e. make $f_{on}(1)$ as large as possible while satisfying some of the other source characteristics (e.g. given values of $\bar{on}$ and/or CV_{on}^2.) One such distribution is the *mixture of two deterministic distributions* which is a distribution that can be of length L_1 or L_2 with probabilities p and $1-p$ respectively. We fix one of the deterministic lengths to be equal to 1 and let the other be of a variable length L. Given a particular value of $\bar{on}$ and ϕ_1 and the off period distribution, we can find values for p and L which would satisfy the given values of $\bar{on}$ and ϕ_1 using a simple enumerative algorithm.

To study the effect of the inter-arrival time correlation on the multiplexer behavior, we consider the case of two input homogeneous sources where the off period of a source has a geometric distribution with mean 92.7787 and the mean length of the on period is fixed at 50, making the source's average rate equal to 0.35. Using the mixture of two deterministic distributions for the on period, we vary ϕ_1 so that it takes values from the set $\{0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45\}$. The value of p and L satisfying the given parameters is then calculated. The cell loss probability and the mean number of cells in the multiplexer queue are shown in figure 11.

We note that by increasing the lag-1 correlation, the cell loss probability and the mean number of cells increase. As it can be seen from figure 11, the cell loss probability increases more sharply than the mean number of cells with the increase of the correlation coefficient. The mean number of cells is almost constant and increases very slowly with the increase of the correlation coefficient.

5.3 Effect of the Squared Coefficient of Variation of the on and off periods on the Multiplexer's Performance



(a) (b)

Figure 12: Effect of the squared coefficient of variation of the on and off periods on multiplexer's performance. Buffer Size = 40, two input on/off sources with mixture-of-geometric on and off periods. (a) Cell loss probability vs. CV_{on}^2. (b) Mean number of cells vs. CV_{on}^2.

Using the mixture-of-geometric distribution we studied the effect of the squared coefficient of variation of the on and off periods, respectively CV_{on}^2 and CV_{off}^2, on the multiplexer's performance. By specifying the mean and squared coefficient of variation of the period length, it is possible to fit a mixture-of-geometric distribution given some conditions are met by the specified mean and coefficient of variation (see [2] for more details).

We considered two input sources each with a average rate fixed at 0.4. The mean on and off periods were fixed at 100 and 150 respectively. CV_{on}^2 and CV_{off}^2 take values from the set $\{1.0, 5.0, 10.0, 15.0.20.0\}$. The results obtained from simulation and detailed superposition are shown in figure 12. Note that by increasing the CV_{on}^2, while CV_{off}^2 is kept constant, both the cell loss probability and mean number of cells increase (see parts (a) and (b) of figure 12). Also, note that the rate of increase of the mean number of cells and cell loss probability when $CV_{on}^2 \in [1, 5]$ is larger than for the rest of the values. Moreover, for larger values of CV_{off}^2, the rate of increase of cell loss probability and mean number of cells as a function of CV_{on}^2 is relatively slower than for smaller values of CV_{off}^2. An interesting result is observed when varying CV_{off}^2 while CV_{on}^2 is kept constant. As CV_{off}^2 increases, the cell loss probability *increases* while the mean number of cells *decreases*.

6 Conclusions

In this paper we presented a new approximation method for characterizing the superposition of multiple independent Markov Renewal Processes. The presentation here focused on discrete-time processes, but the methodology is readily applicable to continuous-time processes with little modification. The model developed for characterizing the superposition is applied to the area of teletraffic engineering and statistical multiplexing. One special case of this model is the arbitrary on/off source that was used in this paper. We have presented a queueing model for the analysis of a statistical multiplexer whose input is a MRP representing the superposition of multiple traffic sources. The methodology is more of theoretical significance since the computation cost is very high for a practical number of processes. However, the purpose of the shown results is to indicate the need for more accurate methods that can capture more traffic parameters that what is currently being used.

The advantage of our methodology is that it provides a uniform framework in which a variety of models of traffic sources can be handled. The basic limitation is the huge state space and the computational complexity of the algorithms.

The work presented here introduces many challenging issues. An important problem that is yet to be considered is the characterization of the departure process from a multiplexer with multiple arbitrary on/off input sources, or in general, with an MRP input. Characterizing the departure process of one of the input sources is even more challenging. The anlysis done here need to be extended to the case where the on/off source has peridoic or probabilistic arrivals.

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